PARAMETRIC EXCITATION OF CORKSCREW INSTABILITY OF A θ-PINCH BY A HIGH-FREQUENCY LONGITUDINAL CURRENT

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Equilibrium conditions are given for a thin annular θ-pinch with high β in a magnetic field, transverse to the plane of the ring, exciting a high-frequency longitudinal current in the plasma. The parametric buildup of small corkscrew perturbations is studied on the basis of the model of a flexible straight plasma filament. Conditions under which parametric excitation may be suppressed are discussed. There is a brief discussion of forced oscillations of the radii of the annular θ-pinch caused by the alternating fields.

When a right circular θ-pinch twists into a torus, a radial repulsive force arises (see, e.g., [1]) which prevents the formation of a closed equilibrium configuration. An interesting possibility of balancing this force and thus achieving an equilibrium toroidal θ-pinch arises when a longitudinal high-frequency current is excited in the plasma and interacts with the external high-frequency magnetic field transverse to the plane of the ring. If the intrinsic pressure of the magnetic field produced by the longitudinal current is much smaller than the plasma pressure, such a discharge may remain of the θ-pinch type. The use of an alternating rather than a direct longitudinal current has the advantage that it significantly relaxes the limitations imposed on the longitudinal current by the condition for stability of the plasma filament with respect to long-wavelength corkscrew perturbations [2,3]. At the same time, a high-frequency longitudinal current may lead to the parametric excitation of a corkscrew instability of the filament at shorter wavelengths [3], and it may lead to forced oscillations of the radii of the annular filament.

1. We consider a thin annular filament with small radius a and large radius R ($a \ll$ R) formed by an ideally conducting compressible plasma. A direct azimuthal current flowing along the surface of the filament governs the jump in the constant longitudinal magnetic field at the plasma–vacuum interface; there is also a high-frequency longitudinal current

$$I = I_0 \cos \omega t \tag{1.1}$$

induced by an external high-frequency magnetic field transverse to the plane of the ring.

The conditions for equilibrium of the ring along its small and large radii are [4]

$$8\pi p + B_i^2 = B_e^2 + \langle B_a^2 \rangle \tag{1.2}$$

$$8\pi p + B_e^2 + (l+2)\langle B_a^2 \rangle = B_i^2 + 4(R/a)\langle B_R B_a \rangle \tag{1.3}$$

where p is the gas pressure, B_e and B_i are the longitudinal magnetic fields outside and inside the filament, $B_a = B_a \cos \omega t = 2I/ca$ is the azimuthal magnetic field of current I at the filament surface, $B_R = B_{R0} \cos \omega t$ is the high-frequency magnetic field at the circle of radius R transverse to the ring, $l = 2\ln(8R/a) - 4$ is the ring inductance per unit length for current I, and the angle brackets denote a time averaging. Strictly speaking, Eqs. (1.2) and (1.3) reflect the fact that oscillations of the small and large ring radii due to the alternating fields occur about the values a and R, respectively.

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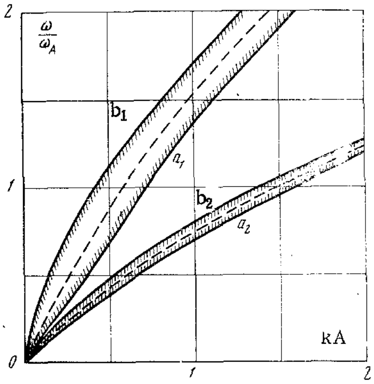

Fig. 1

We restrict the discussion to the case in which this corkscrew discharge is of the θ-pinch type with a nearly unit value of $\beta \equiv 8\pi p/B_e^2$ and we assume that $B_e^2 \approx 8\pi p \gg B_i^2, \langle B_a^2 \rangle$. If, moreover, $B_e^2 \gg l \langle B_a^2 \rangle$ condition (1.3) for equilibrium of the ring along its large radius simplifies considerably, becoming

$$h_a h_R = a / R \tag{1.4}$$

where $h_a = B_{a0}/B_e$, $h_R = B_{R0}/B_e$. It follows from Eq. (1.4) that equilibrium of a thin ring ($a/R \approx 1/100$) can be achieved with $h_R \ll h_a \ll 1$.

In principle, equilibrium confinement of a toroidal θ-pinch can also be acheived through the interaction of an alternating longitudinal current with the conducting sheath around the filament. However, the efficiency is low in this case, since the ratio of the filament radius a to the sheath radius b for θ-pinch discharges is usually very small: $a/b \approx 1/5$.

2. In studying the stability of a plasma filament with respect to corkscrew perturbations, we will for simplicity neglect the toroidal nature of the problem and consider the effect of a transverse magnetic field. In a study of the stability of a thin cylindrical plasma conductor carrying a current with respect to corkscrew perturbations, for which the displacement of the conductor axis is

$$\xi = \xi_1 \exp(ikz \pm i\theta) \tag{2.1}$$

it is convenient to use the model of a flexible filament, calculating the forces exerted by the magnetic field on the perturbed conductor and then directly studying the equation of motion of an arbitrary element of length of the conductor. This approach is particularly effective in studying the stability of a filament with a high-frequency alternating current, since it can be used to study parametric excitation of corkscrew perturbations of the filament. By way of contrast, the method used in [2], involving an averaging over high-frequency oscillations of the pressure-balance equation for the surface of the perturbed conductor – an approach which then yields a dispersion relation – cannot be used for a study of parametric excitation.

The applicability conditions for the equation of motion obtained by this method were rigorously derived in [4, 5] for the general case of quasisteady-state high-frequency magnetic fields for a compressible and ideally conducting plasma filament undergoing smooth ($k\xi \ll 1$), overdeveloped ($\xi \gg a$), and, as a consequence, long-wavelength ($ka \ll 1$) perturbations: $ka \ll k\xi \ll 1$. If it is assumed that for the applicability of this method, as in the case of static fields [6], it is sufficient that only the smoothness condition ($k\xi \ll 1$) be satisfied, short-wavelength perturbations ($ka \gtrsim 1$) may also be studied in this manner, but only if they are underdeveloped ($\xi \ll a$), since inequalities $k\xi \ll 1 \lesssim ka$ must hold. It should be kept in mind, however, that the magnetohydrodynamic description of a plasma is not valid for short-wavelength perturbations in many cases of practical interest.

For the problem as outlined, we introduce an equation for small corkscrew oscillations of a plasma filament carrying longitudinal current (1.1). When there is a perturbation of the form (2.1), the cylindrical plasma filament is acted upon by a force per unit length [7]

$$F = {}^1/_4 [B_a^2 - \alpha_e(ka)(kaB_e \pm B_a)^2 - \alpha_i(ka)(ka)^2 B_i^2]\,\xi_1 \tag{2.2}$$

Here

$$\alpha_e(x) = -K_1(x) / xK_1'(x), \qquad \alpha_i(x) = I_1(x) / xI_1'(x)$$

where $K_1(\chi)$ and $I_1(\chi)$ are modified Bessel functions, and the prime denotes differentiation with respect to the argument.

The equation of motion of an element of the filament length,

$$\pi a^2 \rho \ddot{\xi}_1 = F \tag{2.3}$$

where ρ is the equilibrium plasma density, is an equation with periodic coefficients because of the time dependence adopted for the current I in (1.1). Using the replacement $\omega t = 2\tau$, we can write Eq. (2.3) in the standard form of a Hill equation [8] with the three terms

$$d^2\xi_1 / d\tau^2 + (\theta_0 + 2\theta_1 \cos 2\tau + 2\theta_2 \cos 4\tau)\,\xi_1 = 0 \tag{2.4}$$

where

$$\theta_0 = 4(\omega_A/\omega)^2 [(ka)^2(\alpha_e + \alpha_i h_i^2) + {}^1/_2(\alpha_e - 1) h_a^2]$$

$$\theta_1 = \pm 4(\omega_A/\omega)^2 ka\alpha_e h_a, \quad \theta_2 = (\omega_A/\omega)^2(\alpha_e - 1) h_a^2$$

$$\omega_A = B_e/(4\pi\rho)^{1/2}a, \quad h_i = B_i/B_e$$

In the limiting case $\omega \to \infty$, in which the plasma filament, having a finite inertia, "does not feel" the oscillating components of the force F, the stability condition is governed by the inequality $\theta_0 > 0$ or (for a filament of length L with identified ends) by

$$h_a^2 < 2(1 + h_i^2)/\ln\frac{L}{\eta\pi a} \tag{2.5}$$

where $\ln \eta = 0.577$ ... is the Euler constant. Condition (2.5) does not contradict equilibrium condition (1.4) when $h_R \ll h_a$, so it does not rule out the use of an alternating current along with an alternating transverse field $B_R \ll B_a$ for the equilibrium confinement of a toroidal θ-pinch with high β. On the other hand, the analogous condition in the case of a direct current $I = I_0$,

$$h_a < (1 + h_i^2)\pi a/L, \tag{2.6}$$

makes this confinement method impractical.

The oscillating components of the force F at a finite frequency ω slightly relax criterion (2.5) for the stability of the filament with respect to long-wavelength corkscrew perturbations, but – an important point – they may cause parametric buildup of corkscrew perturbations at shorter wavelengths.

To each periodic term in Eq. (2.4) there corresponds a mechanism for the parametric buildup of oscillations. The first term is due to the force exerted by the external longitudinal magnetic field on the filament with a current as it twists into a helix; the second term is due to the force exerted on the twisted filament by the magnetic field of the current itself. If the alternating longitudinal current is so small that $h_a \ll 1$, then for any ka we have $|\theta_2/\theta_1| < 1/4h_a \ll 1/4$, so that the second of these mechanisms for parametric buildup is much less effective. To simplify the discussion, we will subsequently neglect this mechanism. Instead of (2.4), therefore, we will discuss below the simple equation

$$d^2\xi_1/d\tau^2 + (\theta_0 + 2\theta_1\cos 2\tau)\xi_1 = 0 \tag{2.7}$$

which is the familiar Mathieu equation.

The conditions for the stability of Eq. (2.7) are [8]

$$a_n(\theta_1) < \theta_0 < b_{n+1}(\theta_1) \quad (n = 0, 1, 2, \ldots) \tag{2.8}$$

where $a_n(\theta_1)$, $b_n(\theta_1)$ are the eigenvalues of the Mathieu functions, tabulated in, e.g., [9].

The accompanying figure shows $\theta_0 = a_1, b_1, a_2, b_2$ curves plotted in the coordinates ka, ω/ω_A for the particular case $h_i = 0$, $h_a = 1/4$ (a θ-pinch with an alternating longitudinal current and $\beta \approx 1$). The same curves hold quite accurately for the case $h_i = 1$, $h_a = \sqrt{2}/4$ (a high-frequency z-pinch with $\beta \ll 1$) if the if the quantity $2^{1/2}ka$ is plotted along the abscissa. The hatched regions are those in which parametric excitation of corkscrew perturbations occurs. As $h_a \to 0$, these regions move toward the curves $2ka\alpha_e^{1/2}/n$ (the dashed curves in this figure), which correspond to the condition $\omega = 2\omega_0/n$ for $\omega_0 \equiv {}^1/_2\omega\theta_0^{1/2}$. For $n > 2$, the excitation regions lie below the first two resonance regions show in this figure.

For a filament of finite length L, the parametric excitation regions in this figure separate into several vertical strips corresponding to discrete values of the dimensionless wave number:

$$ka = (2\pi a/L)j \quad (j = 1, 2, 3, \ldots) \tag{2.9}$$

It is convenient to introduce the modulation coefficient $\varepsilon \equiv |2\theta_1/\theta_0|$. For $h_i = 0$, $h_a \ll 1$ (a θ-pinch with an alternating longitudinal current and $\beta \approx 1$), we have $\varepsilon \approx 2h_a/ka = 2\lambda/H$, where $\lambda = 2\pi/k$ is the perturbation wavelength and $H = 2\pi a B_e/B_{a0}$ is the pitch amplitude of the corkscrew magnetic force line at the surface of the equilibrium filaments. For $h_i = 1$, $h_a \ll 1$ (a high-frequency z-pinch with $\beta \ll 1$), we have $\varepsilon \approx h_a/ka = \lambda/H$. Interestingly, the minimum modulation coefficient in this latter case for a filament of finite length is $\varepsilon_{min} = 1/q$, where $9 = H/L$ is the stability-reserve coefficient of the z-pinch, calculated from the amplitude of the azimuthal field.

The parameter ε strongly affects the stability of the system. As an examination will show, the region $\varepsilon > 1$, $\theta_0 > b_1$ is occupied primarily by parametric-excitation regions, while the region $\varepsilon < 1$, $\theta_0 > b_1$ is occupied by stability regions. In the figure, these regions are beneath the curve $\theta_0 = b_1$, to the left and right, respectively, of the vertical line $ka = 0.5$, at which $\varepsilon = 1$.

The modulation coefficient ε decreases with increasing working frequency ω, as shorter and shorter perturbation waves are excited.

It is not difficult to show that we have $\varepsilon < 1$ at any working frequency when condition (2.6) holds. Here, however, the alternating current loses its advantage, which it derives from the possibility of a significant relaxation of condition (2.6).

If $\varepsilon \ll 1$, parametric excitation has a well-defined resonance nature and occurs when ω is equal to or nearly equal to $2\omega_0/n$; oscillations at frequency $1/2\omega n$ are excited in the n-th resonance region. If, on the hand, we have $\varepsilon \gtrsim 1$, the resonance properties of the instability are poorly defined. The oscillations excited in the system have a broad frequency spectrum in this case.

When $\varepsilon \ll 1$, we can find simple analytic expressions for the quantities characterizing the parametric excitation. For example, the relative width $\Delta\omega/2\omega_0$ of the first resonance region and the maximum instability increment γ, reached at $\omega = 2\omega_0$, are in the case of the θ-pinch

$$\begin{aligned} \Delta\omega / 2\omega_0 &= {}^1/_2\varepsilon = h_a / ka \ll 1 \\ \gamma &= {}^1/_8\, \varepsilon\omega = {}^1/_4\, (h_a / ka)\, \omega \ll \omega \end{aligned} \tag{2.10}$$

As n increases, the width of the resonance regions and the instability increment fall off rapidly, in proportion to ε^n [10].

For arbitrary ε and when Eq. (2.5) holds, in the case $\theta_0 > 0$, a sufficient criterion for the stability of the system is, quite accurately, $\theta_0 < 1 - |\theta_1|$. For a θ-pinch, we find

$$\omega / \omega_A > 2ka\,(1 + h_a / ka)^{1/2} \alpha_e^{1/2} \tag{2.11}$$

In the limiting cases, Eq. (2.11) is replaced by

$$\omega / \omega_A > 2ka\,(1 + h_n / ka)^{1/2} \ (ka \ll 1), \qquad \omega / \omega_A > 2\,(ka)^{1/2} \qquad (ka \gg 1) \tag{2.12}$$

The region corresponding to (2.11) is above the curve $\theta_0 = b_1$ in the accompanying figure.

It follows from the stability diagram and Eqs. (2.11) and (2.12) that in the case of a thin filament ($2\pi a/L \ll 1$), for which spectrum (2.9) of dimensionless wave numbers ka is nearly continuous and unbounded, parametric buildup of small corkscrew perturbations may occur at any working frequency alternating current for equilibrium confinement and stabilization [2-4, 11] of plasma in a longitudinal magnetic field. It should be kept in mind, however, that this interpretation is based on an extremely simplified model which does not take into account many factors which may weaken or even completely eliminate parametric excitation.

The effects of the various factors preventing parametric excitation should be displayed first at short-wavelength perturbations, for which the instability has a small increment and is of a clearly resonance nature. If, on the other hand, there is an upper limit on the wave number spectrum of the parametrically excited perturbations, one can stabilize a plasma filament by selecting a sufficiently high working frequency.

Let us first determine the effect of friction in the system on the parametric buildup of corkscrew perturbations. When there is friction, the instability regions contract slightly [10]; if the oscillation damping decrement satisfies $\delta > \gamma$, there will be no parametric excitation.

For the first resonance region with $ka \gg 1$ in the case of a θ-pinch, we have

$$\Delta\omega / 2\omega_0 = [{}^1/_4\,\varepsilon^2 - 4\,(\delta / \omega_0)^2]^{1/2} = [(h_a / ka)^2 - (4 / ka)\,(\delta / \omega_A)^2]^{1/2}$$

We can find the upper limit on the spectrum of parametrically excited corkscrew perturbations:

$$(ka)_{\max} = {}^1/_4\, h_a^2\, (\omega_A / \delta)^2$$

then, using (2.12), we see that the criterion for the filament stability is

$$\omega / \omega_A > h_a (\omega_A / \delta) \tag{2.13}$$

It is easy to see that (2.13) is equivalent to the condition $\delta > \gamma$. Since the instability reaches its maximum increment in the first resonance region, the higher parametric resonances will apparently not occur when criterion (2.13) is satisfied.

Yet another factor which should have a strong effect on the parametric excitation of short-wavelength corkscrew perturbations is the nonsteady-state nature of the actual system. As a result of the smooth time dependence of any system parameter, the excitation condition, of an especially resonant nature at $\varepsilon \ll 1$, will hold only for a time smaller than $1/\gamma$ for excitation with a certain wavelength, since the given perturbation does not manage to build up. Here the excitation energy will be "smeared" over a certain spectral range, never exceeding the danger point.

Where necessary, this effect can be arranged artifically, e.g., by changing the working frequency ω (as was proposed in [12]). If $\omega^{\cdot}$ is chosen to satisfy the conditions

$$1 / \omega \ll |\omega / \omega^{\cdot}| \lesssim 1 / \gamma \tag{2.14}$$

parametric excitation may be suppressed, while the advantages associated with the use of a high-frequency alternating current are retained. Because of (2.10), conditions (2.14) are completely attainable, especially at the most critical (in the sense of the choice of working frequency) short-wavelength perturbations.

3. We will now briefly consider forced oscillations in the radii of an annular θ-pinch under the influence of alternating fields. In particular, we will show that if condition (2.11) holds, even for long-wavelength perturbations, the radial oscillations are stable and of small amplitude. For simplicity, we restrict the discussion to partial oscillations. As in Section 1, we will assume that $h_i^2 \ll 1$, $ha^2 \ll 2/l$, $h_R \ll ha$.

The corresponding equations of motion can be derived through the use of the Routh function given in [4]. Oscillations of the small radius are described by an inhomogeneous Mathieu equation with the coefficients

$$\theta_0 = 4(\omega_A / \omega)^2 \gamma_0 \beta, \qquad \theta_1 = (\omega_A / \omega)^2 h_a^2 (4 - l) / 2l$$

where γ_0 is the ratio of heat capacities, and the right side is equal to

$$-(\omega_A / \omega)^2 h_a^2 \cos 2\tau \qquad (\tau = \omega t)$$

The stability regions of an equation of this type are the same as regions (2.8) of the corresponding homogeneous equation [13]. Since in this case we have $\theta_0 > 0$ and $\varepsilon = ha^2 (l-4)/4\gamma_0 \beta l \ll 1$, the oscillations of the small radius are stable everywhere, except in narrow bands near the fixed frequencies $\omega = 2\omega_A (\gamma_0 \beta)^{1/2}/n$, where parametric excitation of oscillations is in principle possible, but easy to avoid in practice.

Oscillations of the large radius obey an inhomogeneous Mathieu equation with coefficients

$$\theta_0 = 2\theta_1 = (\omega_A / \omega)^2 (a / R)^2 (2 / l h_a^2 - g) \qquad (g = -(R / B_{R0}) (\partial B_{z0} / \partial r)_{r=R})$$

and a right side

$$-(\omega_A / \omega)^2 (a / R)^2 \cos 2\tau \qquad (\tau = \omega t)$$

Since $\varepsilon = 1$ in this case, the conditions $0 < \theta_0 < 1 - |\theta_1|$ must hold for stability. Hence, assuming that $|g| \ll 2/lha^2$, and using (1.4), we find

$$\omega / \omega_A > (3 / l)^{1/2} h_R \tag{3.1}$$

As is easily seen, criterion (3.1) is much weaker than (2.11), even at $ka \approx h_R \ll 1$. Interestingly, under conditions (3.1) and $0 < g \ll 2/lh_a{}^2$, the ring is also stable with respect to small vertical displacements, described by Eq. (2.4) with $\theta_0 = 2\theta_1 = (\omega_A/\omega)^2 (a/R)^2 g$.

The steady-state amplitudes Δa and ΔR of forced oscillations of the ring radii can be evaluated by neglecting the instrinsic elasticity of the filament in the equations of motion. We find

$$\Delta a = {}^1/_4 h_a^2 (\omega_A / \omega)^2 a, \qquad \Delta R = (h_R / h_a) \Delta a$$

We evidently have $\Delta a, \Delta R \ll a$ if condition (2.11) holds, even for $ka \approx h_a \ll 1$.

Within the framework of this approach, therefore, the most rigorous requirement on the frequency of the alternating longitudinal current is condition (2.11), i.e., that there be no parametric excitation of corkscrew filament perturbations.

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